LU Factorization Notes

1. LU Factorization Without Pivoting:

- Want to solve $A\bar{x} = \bar{b}$.
- We can $L_{n-1} L_{n-2} \dots L_1 A = U \Longleftrightarrow A = \underbrace{L_1^{-1} \dots L_{n-1}^{-1}}_{L} U$

where L is a lower triangular matrix and U is an upper triangular matrix.

- Now we have $LU\bar{x} = \bar{b}$.
- Let $U\bar{x} = \bar{d}$
- Now, we solve $L\bar{d} = \bar{b}$ for $\bar{d}$ (forward substitution) and $U\bar{x} = \bar{d}$ for $\bar{x}$ (backward substitution).
- Note: Even if A is non-singular, we may not always be able to use this strategy.
- To compute L_i^{-1}, simply take L_i and switch the sign of the multipliers.
- If L_i is a Gauss Transformation, then L_i^{-1} exists and is also a Gauss Transformation.
- If L_i and L_j are Gauss Transformations, and $j > i$, then $L_i L_j = L_i + L_j - I$

E.g. 1 Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$, find L and U.

Soln

$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

$$L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$L_2 (L_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \leftarrow \text{Upper Triangular Matrix}$$

$$L = L_1^{-1} \cdot L_2^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \leftarrow \text{Lower Triangular Matrix}$$

Another way to compute L in this case is

$L = L_1^{-1} + L_2^{-1} - I$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

Now, let's see what LU equals to.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}}_{A}$$

E.g. 2 Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $\bar{b} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$

Solve using LU factorization.

Soln:

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix}$$

$$L_2(L_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & \frac{13}{3} \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & \frac{13}{3} \end{bmatrix}$$

$$L = L_1^{-1} + L_2^{-1} - I$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -\frac{1}{3} & 1 \end{bmatrix}$$

$LU\bar{x} = \bar{b}$

Let $U\bar{x} = \bar{d}$

Now, we solve $L\bar{d} = \bar{b}$ for $\bar{d}$ and then $U\bar{x} = \bar{d}$ for $\bar{x}$.

$L\bar{d} = \bar{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1/3 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$$

$d_1 = 4$

$d_1 + d_2 = -6$

$4 + d_2 = -6$

$d_2 = -10$

$2d_1 - \frac{d_2}{3} + d_3 = 7$

$6d_1 - d_2 + 3d_3 = 21$

$24 - (-10) + 3d_3 = 21$

$34 + 3d_3 = 21$

$3d_3 = -13$

$d_3 = \frac{-13}{3}$

$$\bar{d} = \begin{bmatrix} 4 \\ -10 \\ -\frac{13}{3} \end{bmatrix}$$

Now, we'll solve $U\bar{x} = \bar{d}$ for $\bar{x}$.

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & \frac{13}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ -\frac{13}{3} \end{bmatrix}$$

$x_3 = -1$

$-3x_2 + 4x_3 = -10$

$-3x_2 - 4 = -10$

$-3x_2 = -6$

$x_2 = 2$

$x_1 + x_2 + x_3 = 4$

$x_1 + 2 + 1 = 4$

$x_1 + 3 = 4$

$x_1 = 1$

$$\bar{x} = \begin{bmatrix} \\ 2 \\ -1 \end{bmatrix} \leftarrow 1$$

2. LU Factorization with Pivoting:

- When we do LU factorization with pivoting, we want the biggest value for each pivot where the value is in the same column as the pivot and is either the pivot or below the pivot.

E.g. Take $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

Look at the second column. The pivot is 1. However, it's the smallest value in that column. We want to replace it with the biggest value in that column s.t. the value is the pivot or below the pivot. In this example, it's 2. We ignore 3 because 3 is above the pivot.

- When we swap/switch rows, we need to multiply by a permutation matrix, P.

- Now, we have $L_{m-1}P_{m-1}\cdots L_2P_2L_1P_1A = U$

$$\iff L_{m-1}\hat{L}_{m-2}\cdots\hat{L}_1 P_{m-1}\cdots P_1 A = U$$

$$\iff \underbrace{P_{m-1}\cdots P_1}_{P} A = \underbrace{\hat{L}_1^{-1}\hat{L}_2^{-1}\cdots L_{m-1}^{-1}}_{L} U$$

$$\iff PA = LU$$

Originally, we had $A\bar{x} = \bar{b}$.
Now, we have $PA\bar{x} = P\bar{b}$

$$\iff LU\bar{x} = P\bar{b}$$

Let $U\bar{x} = \bar{d}$
We solve $\underbrace{L\bar{d} = P\bar{b} \text{ for } \bar{d}}_{\text{Forward Solve}}$ and $\underbrace{U\bar{x} = \bar{d} \text{ for } \bar{x}.}_{\text{Backward Solve}}$

E.g. 3 Solve $\begin{bmatrix} 2 & 6 & 6 \\ 3 & 5 & 12 \\ 6 & 6 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \\ 30 \end{bmatrix}$

using LU factorization with pivoting.

Soln:

Step 1: Since we want the pivot to be the biggest value in the col at or below the pivot, we need to switch rows 1 and 3.

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_1A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 \\ 3 & 5 & 12 \\ 6 & 6 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & 12 \\ 3 & 5 & 12 \\ 2 & 6 & 6 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

$$L_1(P_1A) = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 6 & 12 \\ 3 & 5 & 12 \\ 2 & 6 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & 12 \\ 0 & 2 & 6 \\ 0 & 4 & 2 \end{bmatrix}$$

Step 2: Now, we switch the 2^{nd} and 3^{rd} rows so that the pivot is 4 instead of 2.

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_2(L_1P_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 6 & 12 \\ 0 & 2 & 6 \\ 0 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$L_2(P_2L_1P_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$U = \begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

Step 3: Right now we have $L_2 P_2 L_1 P_1 A$. We want the L's together before the P's. I.e. We want $L_2 \hat{L}_1 P_2 P_1 A$. To do this, we'll multiply $L_2 P_2 L_1 P_1 A$ by $P_2 \cdot P_2$ at a specific spot.

$L_2 P_2 L_1 P_2 P_2 P_1 A$

Note: The inverse of any permutation matrix is itself. So, when we do $P_i \cdot P_i$, we get I.

Note: When we pre-multiply by a permutation matrix, we swap rows. When we post-multiply by a permutation matrix, we swap columns.

$$P_2 L_1 P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 \\ -\frac{1}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \leftarrow \hat{L}_1$$

Note: $\hat{L}_1$ is just L_1 with its multipliers switched.

$L_2 \hat{L}_1 P_2 P_1 A = U$

$$\underbrace{P_2 P_1}_{P} A = \underbrace{\hat{L}_1^{-1} L_2^{-1}}_{L} U$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L = \hat{L}_1^{-1} \cdot L_2^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

Now, instead of $A\bar{x} = \bar{b}$, we have $PA\bar{x} = P\bar{b}$

$\Longleftrightarrow LU\bar{x} = P\bar{b}$

$$P\bar{b} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 20 \\ 25 \\ 30 \end{bmatrix}$$

$$= \begin{bmatrix} 30 \\ 20 \\ 25 \end{bmatrix}$$

$L U \bar{x} = P \bar{b}$

Let $U\bar{x} = \bar{d}$

We solve $\underbrace{L\bar{d} = P\bar{b} \text{ for } \bar{d}}_{\text{Forward Solve}}$ and $\underbrace{U\bar{x} = \bar{d} \text{ for } \bar{x}.}_{\text{Backward Solve}}$

$L\bar{d} = P\bar{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 20 \\ 25 \end{bmatrix}$$

$d_1 = 30$

$\frac{d_1}{3} + d_2 = 20$

$10 + d_2 = 20$

$d_2 = 10$

$\frac{d_1}{2} + \frac{d_2}{2} + d_3 = 25$

$15 + 5 + d_3 = 25$

$d_3 = 5$

$$\bar{d} = \begin{bmatrix} 30 \\ 10 \\ 5 \end{bmatrix}$$

$U\bar{x} = \bar{d}$

$$\begin{bmatrix} 6 & 6 & 12 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \\ 5 \end{bmatrix}$$

$5x_3 = 5$

$x_3 = 1$

$4x_2 + 2x_3 = 10$

$4x_2 + 2 = 10$

$4x_2 = 8$

$x_2 = 2$

$6x_1 + 6x_2 + 12x_3 = 30$

$6x_1 + 12 + 12 = 30$

$6x_1 = 6$

$x_1 = 1$

$$\bar{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$